

On RR couplings on D-branes at order $O(\alpha'^2)$

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Abstract

Recently, it has been found that there are couplings of the RR field strength $F^{(p)}$ and the B-field strength H on the world volume of D_p -branes at order $O(\alpha'^2)$. These couplings which have both world-volume and transverse indices, are invariant under the linear T-duality transformations. Consistency with the nonlinear T-duality indicates that the RR field strength $F^{(p)}$ in these couplings should be replaced by $\mathcal{F}^{(p)} = d\mathcal{C}^{(p-1)}$ where $\mathcal{C} = e^B C$. This replacement, however, produces some non-gauge invariant terms. On the other hand, the nonlinear terms are invariant under the linear T-duality transformations at the level of two B-fields. This allows one to remove some of the nonlinear terms in $\mathcal{F}^{(p)}$. We fix this by comparing the nonlinear couplings with the S-matrix element of one RR and two NSNS vertex operators. Our results indicate that in the expansion of $\mathcal{F}^{(p)}$ one should keep only the B-field gauge invariant terms, *e.g.*, $B \wedge dC^{(p-3)}$ where both indices of B-field lie along the brane. Moreover, in this case one should replace B with $B + 2\pi\alpha' f$ to have the B -field gauge invariance.

1 Introduction

The dynamics of the D-branes of type II superstring theories is well-approximated by the effective world-volume theory which is sum of Dirac-Born-Infeld (DBI) and Chern-Simons (CS) actions. The DBI action which describes the dynamics of the branes in the presence of the NSNS background fields at order $\mathcal{O}(\alpha'^0)$ is given by [1, 2]

$$S_{DBI} = -T_p \int d^{p+1}x e^{-\phi} \sqrt{-\det(G_{ab} + B_{ab})} \quad (1)$$

where G_{ab} and B_{ab} are the pulled back of the bulk fields $G_{\mu\nu}$ and $B_{\mu\nu}$ onto the world-volume of D-brane¹. The abelian gauge field can be added to the action as $B_{ab} \rightarrow B_{ab} + 2\pi\alpha' f_{ab}$. The curvature corrections to this action has been found in [3] by requiring the consistency of the effective action with the $\mathcal{O}(\alpha'^2)$ terms of the corresponding disk-level scattering amplitude [4, 5]. The couplings of non-constant dilaton and B-field at the order $\mathcal{O}(\alpha'^2)$ has been found in [6] by requiring the consistency of the curvature couplings with the standard rules of linear T-duality transformations, and by the scattering amplitude.

The CS part which describes the coupling of the D-branes to the RR potential at order $\mathcal{O}(\alpha'^0)$ is given by [7, 8]

$$S_{CS} = T_p \int_{M^{p+1}} e^B C \quad (2)$$

where M^{p+1} represents the world volume of the D_p -brane, C is the sum over all RR potential forms, *i.e.*, $C = \sum_{n=0}^8 C^{(n)}$, and the multiplication rule is the wedge product. The four form is self-dual, *i.e.*, $dC^{(4)} = *(dC^{(4)})$, and the electric components of the redundant fields $C^{(5)}, \dots, C^{(8)}$ are related to the magnetic components of the RR fields $C^{(0)}, \dots, C^{(3)}$ as $dC^{(8-n)} = *(dC^{(n)})$ for $n = 0, 1, 2, 3$. The abelian gauge field can be added to the action as $B \rightarrow B + 2\pi\alpha' f$. The curvature correction to this action has been found in [9, 10, 11] by requiring that the chiral anomaly on the world volume of intersecting D-branes (I-brane) cancels with the anomalous variation of the CS action. The curvature coupling has been extended in [12, 13] to include the B-field at the order $\mathcal{O}(\alpha'^2)$ by requiring the consistency of the curvature coupling at the order $\mathcal{O}(\alpha'^2)$ with the linear T-duality transformations.

It has been shown in [14] that there are another class of couplings at the order $\mathcal{O}(\alpha'^2)$ which involve the linear RR field strengths $F^{(p)}$, $F^{(p+2)}$ and $F^{(p+4)}$. These couplings have been found by studying the S-matrix element of one RR and one NSNS vertex operators [4]. The couplings for $F^{(p)}$ in the string frame are [14]:

$$S_{CS} \supset \frac{\pi^2 \alpha'^2 T_p}{2!(p-1)!} \int d^{p+1}x \epsilon^{a_0 \dots a_p} \left(F_{ia_2 \dots a_p, a}^{(p)} H_{a_0 a_1}{}^{a, i} - \frac{1}{p} F_{a_1 a_2 \dots a_p, i}^{(p)} (H_{a_0 a}{}^{i, a} - H_{a_0 j}{}^{i, j}) \right) \quad (3)$$

¹Our index conversion is that the Greek letters (μ, ν, \dots) are the indices of the space-time coordinates, the Latin letters (a, d, c, \dots) are the world-volume indices and the letters (i, j, k, \dots) are the normal bundle indices.

where commas denote partial differentiation. These couplings are invariant under the linear T-duality and the B -field gauge transformations. It has been shown in [14] that consistency of the above couplings with nonlinear T-duality requires one to replace the RR field strength $F^{(p)}$ with $\mathcal{F}^{(p)} = d\mathcal{C}^{p-1}$ where $\mathcal{C} = e^B C$. This replacement produces many couplings with nonlinear B -field.

The B -field gauge symmetry, however, is broken by some of the higher orders of B -field. The higher order terms which are coming from expanding \mathcal{F} , are either in terms of field strength H or in terms of B . The gauge symmetry is broken in the latter cases. The symmetry can be restored, however, in the terms in which the two indices of the B -field are world volume, by replacing B with $B + 2\pi\alpha' f$. The terms in which the B -field includes transverse index may be canceled in the full nonlinear T-duality invariant action.

A perturbative method for constructing the nonlinear T-duality invariant action has been introduced in [13]. It works in the following way. Consider the couplings of one RR field and n number of NSNS fields at order $O(\alpha'^2)$. Suppose one makes the coupling at each order of n to be invariant under "linear" T-duality transformations. By "linear" T-duality transformations we mean that the terms with $n = 1$ must be invariant under linear T-duality transformation. However, for the terms with $n > 1$, one must take into account the nonlinear T-duality transformation of the terms with $m < n$ and the linear T-duality transformation of the terms with $m = n$. Then adding all these terms, we expect the final result to be invariant under full nonlinear T-duality transformations at order $O(\alpha'^2)$. As an example consider the DBI action which is invariant under full nonlinear T-duality transformations. Upon expanding it, one finds terms with $n = 1, 2, \dots$, at order $O(\alpha'^0)$. The terms with $n = 1$ are invariant under linear T-duality transformation. The terms with $n = 2$ are not invariant under linear T-duality. However, when one includes the nonlinear T-duality transformations of the $n = 1$ terms, one would find that the result are invariant under T-duality. Similarly for the terms with $n > 2$.

Using the above method, one may add some new terms which are invariant under linear T-duality transformations at the level of two B -fields, to the action (3) to make it invariant under the full nonlinear T-duality transformations. In this way, one can make the resulting couplings to be invariant under the B -field gauge transformations [13]. The gauge invariant couplings corresponding to the first term in (3) at the level of two B -fields are then [14]

$$S_{CS} \supset \pi^2 \alpha'^2 T_p \int d^{p+1}x \epsilon^{a_0 a_1 \dots a_p} H_{a_0 a_1}{}^{a, i} \left(\frac{1}{2!(p-3)!} F_{ia_2 \dots a_{p-2}}^{(p-2)} (B_{a_{p-1} a_p} + 2\pi\alpha' f_{a_{p-1} a_p}) \right. \\ \left. - \frac{1}{3!(p-4)!} C_{a_2 \dots a_{p-3} i}^{(p-3)} H_{a_{p-2} a_{p-1} a_p} + \frac{1}{2!(p-3)!} C_{a_2 \dots a_{p-2}}^{(p-3)} H_{a_{p-1} a_p i} \right)_{,a} \quad (4)$$

where using the above method, the non-gauge invariant term $B_{ia_2} F_{a_3 \dots a_p}^{(p-2)}$ in the expansion of $\mathcal{F}^{(p)}$ whose corresponding coupling is invariant under linear T-duality transformation, has been canceled [13]. The structure of the couplings of one RR potential $C^{(p-3)}$ and

two B-fields which result from the consistency of the Chern-Simons action at order $O(\alpha'^2)$ with the linear T-duality transformations [12, 13], are different from the couplings in (4). In particular the RR potential in those couplings have indices only along the brane. Restricting the RR potential $C^{(p-3)}$ to the cases that it carries transverse indices, there is no couplings for $C^{(p-3)}$ other than those in (4).

In this paper, we would like to confirm the couplings in (4) by the S-matrix method. An outline of the paper is as follows: In section 2.1, using the couplings in (4), we calculate the massless open string pole and the contact terms for the scattering amplitude of one RR and two B-fields. In section 2.2 we examine the calculation of the S-matrix element of one RR and two NSNS vertex operators in superstring theory. In section 2.2.1, we perform the calculation in full details for $C_{ij}^{(2)}$ and expand the amplitude at low energy. We show that there is neither contact term nor massless open string pole at order $O(\alpha'^2)$ which is consistent with the couplings (4). In section 2.2.2, we perform the same calculation for $C_i^{(1)}$. We show that the massless open string pole and the contact terms of the field theory are reproduced exactly by this amplitude at order $O(\alpha'^2)$.

2 Scattering amplitude

A powerful method for finding the low energy field theory of the string theory is to compare the scattering amplitudes in a proposed field theory with the corresponding amplitudes in the string theory at the low energy. The couplings in (3) have been found by studying the scattering amplitude of one RR and one NSNS states. They are extended to the higher order fields in (4) by requiring the consistency of the couplings (3) with nonlinear T-duality. They can be confirmed by the scattering amplitude of one RR and two NSNS states. As we will see, the string theory scattering amplitude at the low energy produces both massless open string and closed string channels as well as some contact terms. The closed string channels dictate the supergravity couplings in the bulk and the couplings of one RR and one NSNS states on the brane. On the other hand, the open string channels and the contact terms dictate only the couplings on the brane in which we are interested in this paper. In the next section, we calculate the massless open string amplitude and the contact terms resulting from the couplings in (4).

2.1 Field theory amplitude

The open string channel in the scattering amplitude of one RR and two B-fields in the field theory is given by the following Feynman amplitude:

$$\mathcal{A}_1^f = V_a(\varepsilon_3, A)G_{ab}(A)V_b(A, \varepsilon_2, \varepsilon_1^{(n)}) + (2 \leftrightarrow 3) \quad (5)$$

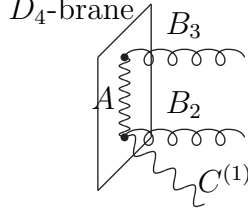


Figure 1 :Feynman diagram for massless open string pole.

where A^a is the gauge field on the D_p -brane. The polarization of the RR field is given by $\varepsilon_1^{(n)}$ and the polarizations of the B-fields are given by $\varepsilon_2, \varepsilon_3$. The on-shell conditions are

$$p_i \cdot p_i = p_i^\mu (\varepsilon_i)_\mu \dots = 0, \quad \text{for } i = 1, 2, 3 \quad (6)$$

To simplify the calculation we restrict the RR polarization tensor to the case that it carries only the transverse indices. Then the only non-zero vertex $V_b(A, \varepsilon_2, \varepsilon_1^{(n)})$ is given by the second term in the first line of (4) for $p = 4$. The vertex is

$$V_b(A, \varepsilon_2, \varepsilon_1^{(1)}) = -2(\pi\alpha')^3 T_4 \epsilon_{a_0 \dots a_3 b} (p_2 \cdot N \cdot \varepsilon_1) (p_2 \cdot V \cdot H_2)^{a_0 a_1} p_1^{a_2} p_2^{a_3} \quad (7)$$

where the matrices $N_{\mu\nu}$ and $V_{\mu\nu}$ project spacetime vectors into transverse and parallel subspace to the D_p -brane, respectively. The gauge field propagator and the vertex $V_a(\varepsilon_3, A)$ can be read from the DBI action (1), *i.e.*,

$$\begin{aligned} V_a(\varepsilon_3, A) &= (2\pi\alpha') T_4 (p_3 \cdot V \cdot \varepsilon_3)_a \\ G_{ab}(A) &= \left(\frac{-i}{T_4 (2\pi\alpha')^2} \right) \frac{\eta_{ab}}{p_3 \cdot V \cdot p_3} \end{aligned} \quad (8)$$

The amplitude then becomes

$$\mathcal{A}_1^f = i \left(\frac{(\pi\alpha')^2 T_4}{3} \right) \frac{p_2 \cdot V \cdot p_2 p_2 \cdot N \cdot \varepsilon_1}{p_3 \cdot V \cdot p_3} \epsilon_{a_0 \dots a_4} (p_3 \cdot V \cdot \varepsilon_3)^{a_4} p_3^{a_3} H_2^{a_0 a_1 a_2} + (2 \leftrightarrow 3) \quad (9)$$

This amplitude is of order $O(\alpha'^2)$ which has six momentum in the numerator and two momentum in the denominator.

There is a contact term at this order which is coming from the first terms in the first and second lines of (4). They are simplify to

$$\mathcal{A}_2^f = -i \frac{(\pi\alpha')^2 T_4}{6} \epsilon_{a_0 \dots a_4} p_2 \cdot N \cdot \varepsilon_1 p_2 \cdot V \cdot p_2 \varepsilon_3^{a_3 a_4} H_2^{a_0 a_1 a_2} + (2 \leftrightarrow 3) \quad (10)$$

Using the following identity:

$$\epsilon_{a_0 \dots a_4} (p_3 \cdot V \cdot H_3)^{a_3 a_4} = \epsilon_{a_0 \dots a_4} (2(p_3 \cdot V \cdot \varepsilon_3)^{a_3} p_3^{a_4} + (p_3 \cdot V \cdot p_3) \varepsilon_3^{a_3 a_4})$$

one can rewrite the sum of \mathcal{A}_1^f and \mathcal{A}_2^f as

$$\mathcal{A}^f = -i \left(\frac{(\pi\alpha')^2 T_4}{6} \right) \frac{p_2 \cdot V \cdot p_2 p_2 \cdot N \cdot \varepsilon_1}{p_3 \cdot V \cdot p_3} \epsilon_{a_0 \dots a_4} (p_3 \cdot V \cdot H_3)^{a_3 a_4} H_2^{a_0 a_1 a_2} + (2 \leftrightarrow 3) \quad (11)$$

Since the result is in terms of the field strengths H_2, H_3 , the amplitude satisfies the Ward identity corresponding to the B-fields. Note that neither the massless open string pole (9) nor the contact term (10) satisfy separately the Ward identity corresponding to the ε_3 . The massless closed string poles in which we are not interested in this paper, should also satisfy the Ward identity corresponding to the B-fields. The amplitude (11), however, does not satisfy the Ward identity corresponding to the RR field, hence, the combination of this amplitude and the massless closed string poles should satisfy the Ward identity associated with the RR field. We will see that the string amplitude which has both open and closed string channels satisfies the Ward identities associated with the RR and B-fields. We now turn to the string theory side and calculate the scattering amplitude of one RR and two B-field vertex operators.

2.2 String theory amplitude

The scattering amplitude of one RR and two NSNS states has been studied in [12] for a particular class of terms in the amplitude to confirm some of the couplings resulting from the consistency of the Chern-Simons action at order $O(\alpha'^2)$ with the linear T-duality transformations. In this paper, however, we are interested in the couplings in (4), so we have to consider a different class of terms in the scattering amplitude. In this section we study the amplitude for the general cases and then focus to the particular class of terms to examine the couplings in (4).

In string theory, the tree level scattering amplitude of one RR and two NSNS states on the world-volume of a D_p -brane is given by the correlation function of their corresponding vertex operators on disk. Since the background charge of the world-sheet with topology of a disk is $Q_\phi = 2$ one has to choose the vertex operators in the appropriate pictures to produce the compensating charge $Q_\phi = -2$. One may choose the RR vertex operator in $(-1/2, -1/2)$ picture, and one of the NSNS vertex operators in $(-1, 0)$ and the other one in $(0, 0)$. However, in this picture the symmetry between the two NSNS is not manifest from the very beginning. After performing the correlators, one has to make more effort to rewrite the final result in a symmetric form. Alternatively, one can choose the RR vertex operator in $(-1/2, -3/2)$ picture [15] and the two NSNS vertex operators in $(0, 0)$ picture. In this form the symmetry of the NSNS states is manifest from the beginning. We prefer to do the calculation in the latter form. We will show that the final result, after using some identities, is independent of the choice of the picture.

The scattering amplitude is given by the following correlation function:

$$\mathcal{A} \sim \langle V_{RR}^{(-1/2, -3/2)}(\varepsilon_1^{(n)}, p_1) V_{NSNS}^{(0,0)}(\varepsilon_2, p_2) V_{NSNS}^{(0,0)}(\varepsilon_3, p_3) \rangle \quad (12)$$

Using the doubling trick [4], the vertex operators are given by the following integrals on the upper half z -plane:²

$$\begin{aligned}
V_{RR}^{(-1/2, -3/2)} &= (P_- H_{1(n)} M_p)^{AB} \int d^2 z_1 : e^{-\phi(z_1)/2} S_A(z_1) e^{ip_1 \cdot X} : e^{-3\phi(\bar{z}_1)/2} S_B(\bar{z}_1) e^{ip_1 \cdot D \cdot X} : \\
V_{NSNS}^{(0,0)} &= (\varepsilon_2 \cdot D)_{\mu_3 \mu_4} \int d^2 z_2 : (\partial X^{\mu_3} + ip_2 \cdot \psi \psi^{\mu_3}) e^{ip_2 \cdot X} : (\partial X^{\mu_4} + ip_2 \cdot D \cdot \psi \psi^{\mu_4}) e^{ip_2 \cdot D \cdot X} : \\
V_{NSNS}^{(0,0)} &= (\varepsilon_3 \cdot D)_{\mu_5 \mu_6} \int d^2 z_3 : (\partial X^{\mu_5} + ip_3 \cdot \psi \psi^{\mu_5}) e^{ip_3 \cdot X} : (\partial X^{\mu_6} + ip_3 \cdot D \cdot \psi \psi^{\mu_6}) e^{ip_3 \cdot D \cdot X} :
\end{aligned}$$

where the indices A, B, \dots are the Dirac spinor indices and $P_- = \frac{1}{2}(1 - \gamma_{11})$ is the chiral projection operator which makes the calculation of the gamma matrices to be with the full 32×32 Dirac matrices of the ten dimensions. The matrix D_ν^μ is diagonal with $+1$ in the world volume directions and -1 in the transverse directions, and

$$\begin{aligned}
H_{1(n)} &= \frac{1}{n!} \varepsilon_{1\mu_1 \dots \mu_n} \gamma^{\mu_1} \dots \gamma^{\mu_n} \\
M_p &= \frac{\pm 1}{(p+1)!} \epsilon_{a_0 \dots a_p} \gamma^{a_0} \dots \gamma^{a_p}
\end{aligned} \tag{13}$$

where ϵ is the volume $(p+1)$ -form of the D_p -brane. It is useful to write the matrix $D_{\mu\nu}$ and the flat metric $\eta_{\mu\nu}$ in terms of the two projection operators $N_{\mu\nu}$ and $V_{\mu\nu}$, *i.e.*,

$$\begin{aligned}
\eta_{\mu\nu} &= V_{\mu\nu} + N_{\mu\nu} \\
D_{\mu\nu} &= V_{\mu\nu} - N_{\mu\nu}
\end{aligned} \tag{14}$$

The components of vectors projected into each of these subspaces N and V or η and D are independent objects. If 1 in the chiral projection P_- produces couplings for $C^{(n)}$, then the γ_{11} produces the couplings for $C^{(10-n)}$. Hence, we consider 1 in the chiral projection and extend the result to all RR potentials.

Choosing the above integral form of the vertex operators, one has to also divide the amplitude (12) by the volume of $SL(2, R)$ group which is the conformal symmetry of the upper half z -plane. We will remove this factor after performing the correlators. Moreover, the overall factor of the amplitude (12) may be fixed by comparing the final result with field theory.

Using the standard world-sheet propagators

$$\begin{aligned}
\langle X^\mu(x) X^\nu(y) \rangle &= -\eta^{\mu\nu} \log(x-y) \\
\langle \psi^\mu(x) \psi^\nu(y) \rangle &= -\frac{\eta^{\mu\nu}}{x-y} \\
\langle \phi(x) \phi(y) \rangle &= -\log(x-y)
\end{aligned} \tag{15}$$

²Our conversions set $\alpha' = 2$ in the string theory calculations.

one can calculate the correlators in (12). The amplitude (12) can be written as

$$\begin{aligned} \mathcal{A} \sim & \frac{1}{2} (H_{1(n)} M_p)^{AB} (\varepsilon_2 \cdot D)_{\mu_3 \mu_4} (\varepsilon_3 \cdot D)_{\mu_5 \mu_6} \int d^2 z_1 d^2 z_2 d^2 z_3 (z_1 - \bar{z}_1)^{-3/4} \\ & \times (b_1 + b_2 + \dots + b_{10})_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} \delta^{p+1} (p_1^a + p_2^a + p_3^a) + (2 \leftrightarrow 3) \end{aligned} \quad (16)$$

where the delta function which gives the conservation of the momenta along the brane is extracted from the X correlators. We have also performed the ghost correlator. The other correlators are

$$\begin{aligned} (b_1)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= \langle : S_A(z_1) : S_B(\bar{z}_1) : \rangle g_1^{\mu_3 \mu_4 \mu_5 \mu_6} \\ (b_2)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= 2(ip_2)_{\beta_1} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \rangle g_2^{\mu_4 \mu_5 \mu_6} \\ (b_3)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= 2(ip_2 \cdot D)_{\beta_1} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_4} : \rangle g_3^{\mu_3 \mu_5 \mu_6} \\ (b_4)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= 2(ip_2)_{\beta_1} (ip_2 \cdot D)_{\beta_2} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \psi^{\beta_2} \psi^{\mu_4} : \rangle g_4^{\mu_5 \mu_6} \\ (b_5)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= (ip_2)_{\beta_1} (ip_3)_{\beta_2} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \psi^{\beta_2} \psi^{\mu_5} : \rangle g_5^{\mu_4 \mu_6} \\ (b_6)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= 2(ip_2)_{\beta_1} (ip_3 \cdot D)_{\beta_2} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \psi^{\beta_2} \psi^{\mu_6} : \rangle g_6^{\mu_4 \mu_5} \\ (b_7)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= (ip_2 \cdot D)_{\beta_1} (ip_3 \cdot D)_{\beta_2} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_4} : \psi^{\beta_2} \psi^{\mu_6} : \rangle g_7^{\mu_3 \mu_5} \\ (b_8)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= 2(ip_2)_{\beta_1} (ip_2 \cdot D)_{\beta_2} (ip_3)_{\beta_3} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \psi^{\beta_2} \psi^{\mu_4} : \psi^{\beta_3} \psi^{\mu_5} : \rangle g_8^{\mu_6} \\ (b_9)_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= 2(ip_2)_{\beta_1} (ip_2 \cdot D)_{\beta_2} (ip_3 \cdot D)_{\beta_3} \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \psi^{\beta_2} \psi^{\mu_4} : \psi^{\beta_3} \psi^{\mu_6} : \rangle g_9^{\mu_5} \\ (b_{10})_{AB}^{\mu_3 \mu_4 \mu_5 \mu_6} &= (ip_2)_{\beta_1} (ip_2 \cdot D)_{\beta_2} (ip_3)_{\beta_3} (ip_3 \cdot D)_{\beta_4} \\ &\quad \times \langle : S_A : S_B : \psi^{\beta_1} \psi^{\mu_3} : \psi^{\beta_2} \psi^{\mu_4} : \psi^{\beta_3} \psi^{\mu_5} : \psi^{\beta_4} \psi^{\mu_6} : \rangle g_{10} \end{aligned} \quad (17)$$

where g 's are the correlators of X 's. Using the propagators (15), one can easily calculate g 's. To find the correlator of ψ , we use the following Wick-like rule for the correlation function involving an arbitrary number of ψ 's and two S 's:

$$\begin{aligned} \langle : S_A(z_1) : S_B(\bar{z}_1) : \psi^{\mu_1}(z_2) \dots \psi^{\mu_n}(z_n) : \rangle = & \quad (18) \\ \frac{1}{2^{n/2}} \frac{(z_1 - \bar{z}_1)^{n/2-5/4}}{\sqrt{(z_2 - z_1)(z_2 - \bar{z}_1)} \dots \sqrt{(z_n - z_1)(z_n - \bar{z}_1)}} \{ & (\gamma^{\mu_n \dots \mu_1} C^{-1})_{AB} + \\ + \mathcal{P}(z_3, z_2) \eta^{\mu_2 \mu_1} (\gamma^{\mu_n \dots \mu_3} C^{-1})_{AB} + \mathcal{P}(z_3, z_2) \mathcal{P}(z_5, z_4) \eta^{\mu_2 \mu_1} \eta^{\mu_4 \mu_3} & (\gamma^{\mu_n \dots \mu_5} C^{-1})_{AB} \\ + \dots \pm \text{perms} \} & \end{aligned}$$

where dots mean sum over all possible contractions. In above equation, $\gamma^{\mu_n \dots \mu_1}$ is the totally antisymmetric combination of the gamma matrices and $\mathcal{P}(z_i, z_j)$ is given by the Wick-like contraction

$$\mathcal{P}(z_i, z_j) \eta^{\mu\nu} = [\psi^\mu(z_i), \widehat{\psi^\nu(z_j)}] = \eta^{\mu\nu} \frac{(z_i - z_1)(z_j - \bar{z}_1) + (z_j - z_1)(z_i - \bar{z}_1)}{(z_i - z_j)(z_1 - \bar{z}_1)} \quad (19)$$

The Wick-like rule has been found in [16] for the case that ψ 's are set at different points in the real axis. It has been extended in [17] to the case that ψ 's in the real axis appear

as currents, *i.e.*, two ψ 's appear in one point. In that case, the only subtlety in using the Wick-like rule is that one must not consider the Wick-like contraction for the two ψ 's in one current. In above we have extended further the formula to the case that the ψ 's are in the complex plane. See [18, 19], for the correlation function of ψ 's with four and more spin operators.

Combining the gamma matrices coming from the correlation (18) with the gamma matrices in (16), one finds the following trace:

$$\begin{aligned} T(n, p, m) &= (H_{1(n)} M_p)^{AB} (\gamma^{\alpha_1 \dots \alpha_m} C^{-1})_{AB} A_{[\alpha_1 \dots \alpha_m]} \\ &= \frac{1}{n!(p+1)!} \varepsilon_{1\nu_1 \dots \nu_n} \epsilon_{a_0 \dots a_p} A_{[\alpha_1 \dots \alpha_m]} \text{Tr}(\gamma^{\nu_1} \dots \gamma^{\nu_n} \gamma^{a_0} \dots \gamma^{a_p} \gamma^{\alpha_1 \dots \alpha_m}) \end{aligned} \quad (20)$$

where $A_{[\alpha_1 \dots \alpha_m]}$ is an antisymmetric combination of the momenta and/or the polarizations of the NSNS states. For example the correlator b_{10} in (17) produces terms with $m = 0, 2, 4, 6, 8$. For $m = 8$ it is

$$A_{[\alpha_1 \dots \alpha_8]} = (\varepsilon_3 \cdot D)_{[\alpha_3 \alpha_1} (\varepsilon_2 \cdot D)_{\alpha_7 \alpha_5} (ip_2)_{\alpha_8} (ip_2 \cdot D)_{\alpha_6} (ip_3)_{\alpha_4} (ip_3 \cdot D)_{\alpha_2]} \quad (21)$$

The trace (20) can be evaluated for specific values of n . One can verify that the amplitude is non-zero only for $n = p - 3, n = p - 1, n = p + 1, n = p + 3, n = p + 5$.

To examine the couplings in (4) with the S-matrix element (12), we consider in this paper only the case

$$n = p - 3 \quad (22)$$

In this case, the trace relation (20) gives non-zero result only for $m \geq 4$. One immediately concludes that b_1, b_2 and b_3 in (17) have no contribution to the amplitude. There are still a lot of terms with non-zero contributions. For ease of calculation we restrict the RR polarization tensor to some specific cases as in the field theory calculation. We consider the cases that the non-vanishing components of the RR polarization to be ε_1^{ij} and ε_1^i . Let us begin with the RR polarization ε_1^{ij} .

2.2.1 ε_1^{ij} case

In this case $n = 2$, and from the relation (22) one gets $p = 5$. Since the indices of the RR potential are transverse and the indices of the volume form are the world-volume indices, the trace (20) is non-zero only for $m = 8$. The trace for $m = 8$ becomes

$$T(2, 5, 8) = 32 \frac{8!}{2!6!} \varepsilon_1^{ij} \epsilon^{a_0 \dots a_5} A_{[ija_0 \dots a_5]} \quad (23)$$

where 32 is the trace of the 32×32 identity matrix.

Since $m = 8$, only b_{10} has non-zero contribution to the amplitude (12). The X correlator in b_{10} is

$$g_{10} = |z_{12}|^{2p_1 \cdot p_2} |z_{13}|^{2p_1 \cdot p_3} |z_{23}|^{2p_2 \cdot p_3} |z_{1\bar{2}}|^{2p_1 \cdot D \cdot p_2} |z_{1\bar{3}}|^{2p_1 \cdot D \cdot p_3} |z_{2\bar{3}}|^{2p_2 \cdot D \cdot p_3} \\ \times (z_{1\bar{1}})^{p_1 \cdot D \cdot p_1} (z_{2\bar{2}})^{p_2 \cdot D \cdot p_2} (z_{3\bar{3}})^{p_3 \cdot D \cdot p_3} (i)^{p_1 \cdot D \cdot p_1 + p_2 \cdot D \cdot p_2 + p_3 \cdot D \cdot p_3} \equiv K \quad (24)$$

where $z_{ij} = z_i - z_j$ and $z_{i\bar{j}} = z_i - \bar{z}_j$. We have added the phase factor to make it real. The above function appears in all other X correlators in (17).

Replacing in (16) the above X -correlator and the ψ -correlator from (18), one finds

$$\mathcal{A} \sim \frac{1}{2} \frac{8!}{6!} \varepsilon_1^{ij} \epsilon^{a_0 \dots a_5} A_{[ija_0 \dots a_5]} \mathcal{I} \quad (25)$$

where $A_{[ija_0 \dots a_5]}$ is given in (21). The conservation of momentum is understood in the above and in all other amplitudes in this paper. The integral in the above amplitude is

$$\mathcal{I} = \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{(z_{1\bar{1}})^2 K}{|z_{12}|^2 |z_{13}|^2 |z_{2\bar{1}}|^2 |z_{3\bar{1}}|^2} \quad (26)$$

which is real as expected. To simplify the kinematic factor in (25), we note that the indices of the RR polarization and the volume form are each antisymmetric. Hence, there are 28 different terms, however, using the definition of matrix D

$$D^{\mu i} = -\delta^{\mu i}, \quad \text{and} \quad D^{\mu a} = \delta^{\mu a} \quad (27)$$

24 of them are zero, and the rest are identical. The amplitude (25) then becomes

$$\mathcal{A} \sim 4(\varepsilon_1)_{ij} \epsilon_{a_0 \dots a_5} p_2^{a_0} p_3^{a_1} p_2^i p_3^j \varepsilon_2^{a_2 a_3} \varepsilon_3^{a_4 a_5} \mathcal{I} + (2 \leftrightarrow 3) \quad (28)$$

It can be seen that the amplitude satisfies the Ward identities corresponding to the B -fields. Using the fact that the only non-zero components of the RR polarization is $(\varepsilon_1)_{ij}$ one observes that the amplitude also satisfies the Ward identity corresponding to the RR gauge field, *i.e.*, the amplitude can be written in terms of the RR field strength $F_{a_0 ij}$.

We now calculate the integral (26). Using the conservation of the momentum, *i.e.*,

$$(p_1 + p_2 + p_3) \cdot V_\mu = 0 \quad (29)$$

one can easily check that the integrand is invariant under the $SL(2, R)$ transformation. So we can map the results to disk with unit radius. The map from upper half plane to unit disk is

$$z \rightarrow -i \frac{z - 1}{z + 1} \quad (30)$$

Under this map, one has

$$\begin{aligned}
z_{i\bar{j}} &\rightarrow -\frac{-2i}{(z_i+1)(\bar{z}_j+1)}(1-z_i\bar{z}_j) \\
z_{ij} &\rightarrow \frac{-2i}{(z_i+1)(z_j+1)}z_{ij} \\
d^2z_i &\rightarrow \frac{4}{(z_i+1)(\bar{z}_i+1)}d^2z_i
\end{aligned} \tag{31}$$

The factors like $\frac{-2i}{(z_i+1)(\bar{z}_j+1)}$ are canceled because the integrand is $SL(2, R)$ invariant. So to map the integral (26) to unit disk one has to use the following replacement:

$$\begin{aligned}
z_{i\bar{j}} &\rightarrow -(1-z_i\bar{z}_j) \\
z_{ij} &\rightarrow z_{ij} \\
z_{i\bar{j}} &\rightarrow -z_{i\bar{j}} \\
z_{i\bar{j}} &\rightarrow (1-\bar{z}_iz_j)
\end{aligned}$$

Obviously the result is still $SL(2, R)$ invariant. To fix this symmetry, we then set [20]

$$z_1 = 0, \quad \text{and} \quad z_2 = \bar{z}_2 = r_2 \tag{32}$$

Under this fixing the measure changes as

$$d^2z_1 d^2z_2 d^2z_3 \rightarrow r_2 dr_2 r_3 dr_3 d\theta, \quad 0 < r_2, r_3 < 1, 0 < \theta < 2\pi \tag{33}$$

where we have chosen the polar coordinate $z_3 = r_3 e^{i\theta}$. The integral (26) then becomes

$$\mathcal{I} = \int_0^1 dr_2 \int_0^1 dr_3 \int_0^{2\pi} d\theta \frac{\widetilde{K}}{r_2 r_3} \tag{34}$$

where \widetilde{K} is

$$\begin{aligned}
\widetilde{K} &= r_2^{2p_1 \cdot p_2} r_3^{2p_1 \cdot p_3} (1-r_2^2)^{p_2 \cdot D \cdot p_2} (1-r_3^2)^{p_3 \cdot D \cdot p_3} \\
&\times |r_2 - r_3 e^{i\theta}|^{2p_2 \cdot p_3} |1 - r_2 r_3 e^{i\theta}|^{2p_2 \cdot D \cdot p_3}
\end{aligned} \tag{35}$$

By looking at \widetilde{K} , one realizes that there are 6 independent Mandelstam variables in the general case of three closed string amplitude, *i.e.*,

$$p_1 \cdot p_2, p_1 \cdot p_3, p_2 \cdot p_3, p_2 \cdot D \cdot p_2, p_3 \cdot D \cdot p_3, p_2 \cdot D \cdot p_3 \tag{36}$$

This number is not the same as the physical open or closed string channels. There are 4 closed string channels $p_1 \cdot p_2, p_1 \cdot p_3, p_2 \cdot p_3$ and $(p_1 + p_2 + p_3)^2$, and there are 3 open string

channels $p_1 \cdot D \cdot p_1$, $p_2 \cdot D \cdot p_2$ and $p_3 \cdot D \cdot p_3$. The physical channels $(p_1 + p_2 + p_3)^2$ and $p_1 \cdot D \cdot p_1$ can be written in terms of the above independent variables as

$$\begin{aligned} (p_1 + p_2 + p_3)^2 &= 2p_1 \cdot p_2 + 2p_1 \cdot p_3 + 2p_2 \cdot p_3 \\ p_1 \cdot D \cdot p_1 &= p_2 \cdot D \cdot p_2 + p_3 \cdot D \cdot p_3 + 2p_2 \cdot p_3 + 2p_2 \cdot D \cdot p_3 \end{aligned} \quad (37)$$

There is no open or closed string channel corresponding to the Mandelstam variable $p_2 \cdot D \cdot p_3$.

In the disk level amplitude of the color-ordered N open string states, there are $N(N-3)/2$ independent Mandelstam variables and there are the same number of open string channels. However, not all the Mandelstam variables in the physical open string channels are the independent variables that appear naturally in the amplitude. There are Mandelstam variables for which there are no open string channel. Hence, one may set them to zero to simplify the calculations. Using this observation, the scattering amplitude of N -point function has been studied in a restricted kinematic setup in [21]. This makes the calculation of the integrals involved to be much easier to perform.

To make easier the calculation of the integral over the θ -coordinate in (34), we here also work in a restricted kinematic setup. Examining the Feynman diagrams involved, one can easily verify that the amplitude considered in this paper has no massless pole in the $p_2 \cdot p_3$ -channel. On the other hand the kinematic factor $\varepsilon_1^{ij} \epsilon^{a_0 \dots a_5} A_{[ij a_0 \dots a_5]}$ does not have any term proportional to $p_2 \cdot p_3$. That means the integral in (34) has no $1/(p_2 \cdot p_3)$ pole. Moreover, there is no closed or open string channel corresponding to the Mandelstam variable $p_2 \cdot D \cdot p_3$, hence, it is safe to restrict the Mandelstam variables in (35) to

$$p_2 \cdot D \cdot p_3 = 0, \quad \text{and} \quad p_2 \cdot p_3 = 0 \quad (38)$$

Let us first consider only the constraint $p_2 \cdot p_3 = 0$. This makes the integral over θ in (34) to be

$$\begin{aligned} \mathcal{I} &= 2\pi \int_0^1 dr_2 \int_0^1 dr_3 r_2^{2p_1 \cdot p_2 - 1} r_3^{2p_1 \cdot p_3 - 1} (1 - r_2^2)^{p_2 \cdot D \cdot p_2} (1 - r_3^2)^{p_3 \cdot D \cdot p_3} \\ &\quad \times {}_2F_1 \left[\begin{matrix} -p_2 \cdot D \cdot p_3, & -p_2 \cdot D \cdot p_3 \\ 1 \end{matrix} ; r_2^2 r_3^2 \right] \end{aligned} \quad (39)$$

where we have used the following relation [23]:

$$\int_0^{2\pi} d\theta \frac{\cos(n\theta)}{(1 + a^2 - 2a \cos(\theta))^b} = 2\pi a^n \frac{\Gamma(b+n)}{n! \Gamma(b)} {}_2F_1 \left[\begin{matrix} b, & n+b \\ n+1 \end{matrix} ; a^2 \right] \quad (40)$$

where $|a| < 1$. Then using the integral representation of the generalized hypergeometric function ${}_pF_q$ (see Appendix), one finds that the integral over r_2 and r_3 gives

$$\begin{aligned} \mathcal{I} &= \frac{\pi}{2} B(p_1 \cdot p_2, 1 + p_2 \cdot D \cdot p_2) B(p_1 \cdot p_3, 1 + p_3 \cdot D \cdot p_3) \\ &\quad \times {}_4F_3 \left[\begin{matrix} p_1 \cdot p_2, & p_1 \cdot p_3, & -p_2 \cdot D \cdot p_3 & -p_2 \cdot D \cdot p_3 \\ 1 + p_1 \cdot p_2 + p_2 \cdot D \cdot p_2, & 1 + p_1 \cdot p_3 + p_3 \cdot D \cdot p_3, & 1 \end{matrix} ; 1 \right] \end{aligned}$$

As we have anticipated above, there is no closed or open string channels in $p_2 \cdot D \cdot p_3$. So it is consistent to use the constraint (38) under which the above integral simplifies to

$$\mathcal{I} = \frac{\pi}{2} B(p_1 \cdot p_2, 1 + p_2 \cdot D \cdot p_2) B(p_1 \cdot p_3, 1 + p_3 \cdot D \cdot p_3) \quad (41)$$

To study the low energy limit of the amplitude (28), we expand \mathcal{I} at the low energy, *i.e.*,

$$\mathcal{I} = \frac{\pi}{2} \left(\frac{1}{p_1 \cdot p_2 p_1 \cdot p_3} - \frac{\pi^2}{6} \left[\frac{p_2 \cdot D \cdot p_2}{p_1 \cdot p_3} + \frac{p_3 \cdot D \cdot p_3}{p_1 \cdot p_2} \right] + \dots \right) \quad (42)$$

The first term is a double massless closed string pole and the second terms are simple massless closed string poles. The dots in the above expansion include terms with four and higher order of momenta. We have seen in the field theory calculation in the previous section that there is no massless open string channel for ε_1^{ij} . This is consistent with the expansion of (42) which has no massless open string channel.

Before leaving this section, we note that the double pole may seem to be unphysical pole, *i.e.*, there is no Feynman diagram corresponding to this pole. However, one can rearrange it in terms of the physical massless closed string double pole as

$$\frac{1}{p_1 \cdot p_2 p_1 \cdot p_3} = \frac{1}{p_1 \cdot p_2 (p_1 \cdot p_2 + p_1 \cdot p_3)} + \frac{1}{p_1 \cdot p_3 (p_1 \cdot p_2 + p_1 \cdot p_3)} \quad (43)$$

and the pole $1/(p_1 \cdot p_2 + p_1 \cdot p_3)$ is in fact $2/(p_1 + p_2 + p_3)^2$ in which the constraint (38) has been used. Similar decomposition can be done for massive poles of the amplitude (28). So the amplitude (28) has three closed string channels $p_1 \cdot p_2$, $p_1 \cdot p_3$, $(p_1 \cdot p_2 + p_1 \cdot p_3)$ and three open string channels $p_2 \cdot D \cdot p_2$, $p_3 \cdot D \cdot p_3$, $p_1 \cdot D \cdot p_1$, however, there are no massless poles in the latter channels.

2.2.2 ε_1^i case

In this case $n = 1$, and from the relation (22) one finds $p = 4$. Since the index of the RR potential is transverse and the indices of the volume form are the world-volume indices, one finds that the trace relation (20) is non-zero only for $m = 6$. The trace in this case becomes

$$T(1, 4, 6) = 32 \frac{6!}{5!} \varepsilon_1^i \epsilon^{a_0 \dots a_4} A_{[ia_0 \dots a_4]} \quad (44)$$

Since $m = 6$, the ψ correlators in b_8 , b_9 and b_{10} have non-zero contributions. For b_8 , b_9 the first term in (18) and for b_{10} the second terms in (18) have non-zero contributions. The X correlators in b_8 , b_9 , b_{10} are

$$g_8^{\mu_6} = i \left(\frac{p_1^{\mu_6}}{z_{1\bar{3}}} + \frac{p_2^{\mu_6}}{z_{2\bar{3}}} + \frac{p_3^{\mu_6}}{z_{3\bar{3}}} + \frac{(p_1 \cdot D)^{\mu_6}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_6}}{z_{2\bar{3}}} \right) K$$

$$\begin{aligned}
g_9^{\mu_5} &= i \left(\frac{p_1^{\mu_5}}{z_{13}} + \frac{p_2^{\mu_5}}{z_{23}} + \frac{(p_3 \cdot D)^{\mu_5}}{z_{33}} + \frac{(p_1 \cdot D)^{\mu_5}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_5}}{z_{2\bar{3}}} \right) K \\
g_{10} &= K
\end{aligned} \tag{45}$$

Where K is given in (24). Using the on-shell condition $\varepsilon_2 \cdot p_2 = \varepsilon_3 \cdot p_3 = 0$ and conservation of momentum along the world volume, *i.e.*,

$$p_1 + p_1 \cdot D + p_2 + p_2 \cdot D + p_3 + p_3 \cdot D = 0 \tag{46}$$

one can rewrite g_8, g_9 as

$$\begin{aligned}
g_8^{\mu_6} &= \frac{iK}{z_{3\bar{3}}} \left(\frac{p_1^{\mu_6} z_{31}}{z_{1\bar{3}}} + \frac{p_2^{\mu_6} z_{32}}{z_{2\bar{3}}} + \frac{(p_1 \cdot D)^{\mu_6} z_{3\bar{1}}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_6} z_{3\bar{2}}}{z_{2\bar{3}}} \right) \\
g_9^{\mu_5} &= \frac{iK}{z_{3\bar{3}}} \left(\frac{p_1^{\mu_5} z_{31}}{z_{13}} + \frac{p_2^{\mu_5} z_{32}}{z_{23}} + \frac{(p_1 \cdot D)^{\mu_5} z_{3\bar{1}}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_5} z_{3\bar{2}}}{z_{2\bar{3}}} \right)
\end{aligned}$$

The reason for this arrangement is that all terms behave similarly under the $SL(2, R)$ transformation. Writing the sub-amplitudes \mathcal{A}_i in (16) corresponding to b_i , the sub-amplitudes corresponding to b_8, b_9 are

$$\begin{aligned}
\mathcal{A}_8 &\sim 4 \frac{6!}{5!} \varepsilon_1^i \varepsilon^{a_0 \dots a_4} (D \cdot \varepsilon_3^T)_{\mu_6 [i} (\varepsilon_2 \cdot D)_{a_3 a_1} (p_2)_{a_4} (p_2 \cdot D)_{a_2} (p_3)_{a_0]} \\
&\quad \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}} K}{|z_{21}|^2 |z_{2\bar{1}}|^2 z_{31} z_{3\bar{1}} z_{3\bar{3}}} \left(\frac{p_1^{\mu_6} z_{31}}{z_{1\bar{3}}} + \frac{p_2^{\mu_6} z_{32}}{z_{2\bar{3}}} + \frac{(p_1 \cdot D)^{\mu_6} z_{3\bar{1}}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_6} z_{3\bar{2}}}{z_{2\bar{3}}} \right) \\
\mathcal{A}_9 &\sim 4 \frac{6!}{5!} \varepsilon_1^i \varepsilon^{a_0 \dots a_4} (\varepsilon_3 \cdot D)_{\mu_5 [i} (\varepsilon_2 \cdot D)_{a_3 a_1} (p_2)_{a_4} (p_2 \cdot D)_{a_2} (p_3 \cdot D)_{a_0]} \\
&\quad \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}} K}{|z_{21}|^2 |z_{2\bar{1}}|^2 z_{31} z_{3\bar{1}} z_{3\bar{3}}} \left(\frac{p_1^{\mu_5} z_{31}}{z_{13}} + \frac{p_2^{\mu_5} z_{32}}{z_{23}} + \frac{(p_1 \cdot D)^{\mu_5} z_{3\bar{1}}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_5} z_{3\bar{2}}}{z_{2\bar{3}}} \right)
\end{aligned}$$

To calculate the kinematic factors, we note that there are 6 different terms, however, 4 of them are zero and the other two terms are equal. Using $D_{\mu a} = \delta_{\mu a}$, the amplitudes then become

$$\begin{aligned}
\mathcal{A}_8 &\sim -8 \varepsilon_1^i \varepsilon^{a_0 \dots a_4} (D \cdot \varepsilon_3^T)_{\mu_6 a_3} (\varepsilon_2)_{a_1 a_4} (p_2)_i (p_2)_{a_2} (p_3)_{a_0} \\
&\quad \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}} K}{|z_{21}|^2 |z_{2\bar{1}}|^2 z_{31} z_{3\bar{1}} z_{3\bar{3}}} \left(\frac{p_1^{\mu_6} z_{31}}{z_{1\bar{3}}} + \frac{p_2^{\mu_6} z_{32}}{z_{2\bar{3}}} + \frac{(p_1 \cdot D)^{\mu_6} z_{3\bar{1}}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_6} z_{3\bar{2}}}{z_{2\bar{3}}} \right) \\
\mathcal{A}_9 &\sim -8 \varepsilon_1^i \varepsilon^{a_0 \dots a_4} (\varepsilon_3)_{\mu_5 a_3} (\varepsilon_2)_{a_1 a_4} (p_2)_i (p_2)_{a_2} (p_3)_{a_0} \\
&\quad \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}} K}{|z_{21}|^2 |z_{2\bar{1}}|^2 z_{31} z_{3\bar{1}} z_{3\bar{3}}} \left(\frac{p_1^{\mu_5} z_{31}}{z_{13}} + \frac{p_2^{\mu_5} z_{32}}{z_{23}} + \frac{(p_1 \cdot D)^{\mu_5} z_{3\bar{1}}}{z_{1\bar{3}}} + \frac{(p_2 \cdot D)^{\mu_5} z_{3\bar{2}}}{z_{2\bar{3}}} \right)
\end{aligned} \tag{47}$$

There are similar sub-amplitudes as above in the $(2 \leftrightarrow 3)$ part of the amplitude (16). The above amplitudes are zero for symmetric polarization tensor ε_2 . It produces terms which

contain $(p\varepsilon_3)_\mu$. In the $p_1 \cdot V \cdot \varepsilon_3$ part of the above amplitudes, one can use the conservation of momentum along the brane (29) to write them in terms of $p_2 \cdot V \cdot \varepsilon_3$ and $p_2 \cdot V \cdot \varepsilon_3$. We will see shortly that there are other $p_2 \varepsilon_3$ and $p_3 \varepsilon_3$ contributions from b_{10} . However, there is no contribution to the $p_1 \cdot N \cdot \varepsilon_3$, $p_2^i p_2 \cdot N \cdot \varepsilon_3$ and $p_2^i p_2 \cdot V \cdot \varepsilon_3$ from b_{10} . So let us consider these structures in the above amplitudes.

The $p_1 \cdot N \cdot \varepsilon_3$ structure of the amplitude is zero for symmetric polarization tensor ε_3 , and for antisymmetric tensor ε_3 it is

$$\mathcal{A}(p_1 \cdot N \cdot \varepsilon_3) \sim -16\varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_1 \cdot N \cdot \varepsilon_3)_{a_3} (\varepsilon_2)_{a_1 a_4} (p_2)_i (p_2)_{a_2} (p_3)_{a_0} \mathcal{I}_1 \quad (48)$$

where

$$\mathcal{I}_1 = \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}} K}{|z_{21}|^2 |z_{2\bar{1}}|^2 z_{3\bar{3}}} \left(\frac{1}{z_{1\bar{3}} z_{3\bar{1}}} - \frac{1}{z_{1\bar{3}} z_{31}} \right)$$

The integrand is real as expected. One can easily verify that the integral is the one appears in the previous section, *i.e.*,

$$\mathcal{I}_1 = -\mathcal{I} \quad (49)$$

The $p_2^i p_2 \cdot N \cdot \varepsilon_3$ structure is

$$\mathcal{A}(p_2 \cdot N \cdot \varepsilon_3, p_2 \cdot N \cdot \varepsilon_1) \sim 8\varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_2 \cdot N \cdot \varepsilon_3)_{a_3} (\varepsilon_2)_{a_1 a_4} (p_2)_i (p_2)_{a_2} (p_3)_{a_0} \mathcal{I}_2 \quad (50)$$

where

$$\mathcal{I}_2 = \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}} (|z_{32}|^2 - |z_{\bar{3}2}|^2) K}{|z_{21}|^2 |z_{2\bar{1}}|^2 z_{3\bar{3}}} \left(\pm \frac{1}{z_{31} z_{3\bar{1}} z_{2\bar{3}} z_{23}} + \frac{1}{z_{\bar{3}1} z_{\bar{3}\bar{1}} z_{23} z_{2\bar{3}}} \right)$$

where the plus sign is for antisymmetric tensor ε_3 and the minus sign is for the symmetric tensor. The integrand is pure imaginary for symmetric and is real for antisymmetric case. The $p_2^i p_2 \cdot V \cdot \varepsilon_3$ structure is

$$\mathcal{A}(p_2 \cdot V \cdot \varepsilon_3, p_2 \cdot N \cdot \varepsilon_1) \sim -8\varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_2 \cdot V \cdot \varepsilon_3)_{a_3} (\varepsilon_2)_{a_1 a_4} (p_2)_i (p_2)_{a_2} (p_3)_{a_0} \mathcal{I}_3 \quad (51)$$

where

$$\mathcal{I}_3 = \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}} K}{|z_{21}|^2 |z_{2\bar{1}}|^2 z_{3\bar{3}}} \left(\frac{1 \pm 1}{z_{1\bar{3}} z_{3\bar{1}}} + \frac{1 \pm 1}{z_{1\bar{3}} z_{31}} \pm \frac{|z_{32}|^2 + |z_{\bar{3}2}|^2}{z_{31} z_{3\bar{1}} z_{2\bar{3}} z_{23}} + \frac{|z_{32}|^2 + |z_{\bar{3}2}|^2}{z_{\bar{3}1} z_{\bar{3}\bar{1}} z_{23} z_{2\bar{3}}} \right)$$

where the plus sign is for antisymmetric and minus sign is for the symmetric tensor ε_3 . The integrand is pure imaginary for symmetric and is real for antisymmetric case. The RR tensor in above amplitudes can be written in terms of $F_{a_0 i}$, so they are invariant under the RR gauge transformation. The ε_2 can also be written in terms of the field strength. So they satisfy the Ward identity associated with ε_2 . However, they do not satisfy the Ward

identity associated with ε_3 . We will see later that the combination of these terms and some other terms in b_{10} satisfy the Ward identity.

Now let us calculate the \mathcal{A}_{10} part of the amplitude. The ψ correlators (18) for b_{10} gives 24 different contractions. There are 6 contractions in which two indices of the NSNS polarization tensors contract with each other, *e.g.*, $(\varepsilon_2 \cdot D \cdot \varepsilon_3)_{\mu_3 \mu_5}$. They give zero result because three indices of the factor $(ip_2)_{\beta_1} (ip_2 \cdot D)_{\beta_2} (ip_3)_{\beta_3} (ip_3 \cdot D)_{\beta_4}$ must then contract with the world-volume tensor which is zero. There are 6 contractions where two momenta contract with each other, *e.g.*, $p_2 D p_2$. There are 6 contractions which produce structure $p \varepsilon_3$. These terms have similar structure as terms in (47). There are another 6 contractions which produce structure $p \varepsilon_2$. They have similar structure as the $(2 \leftrightarrow 3)$ partner of (47). The contractions which produce structure $p \varepsilon_3$ give the following contribution to the amplitude:

$$\begin{aligned} \mathcal{A}_{10}(p \varepsilon_3) \sim & 2 \frac{6!}{5!} \varepsilon_1^i \epsilon^{a_0 \dots a_4} \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}}^2 K}{|z_{21}|^2 |z_{31}|^2 |z_{2\bar{1}}|^2 |z_{3\bar{1}}|^2} \\ & \left(\mathcal{P}(z_2, z_3) A_{1[ia_0 \dots a_4]} + \mathcal{P}(\bar{z}_2, z_3) A_{2[ia_0 \dots a_4]} + \mathcal{P}(z_3, \bar{z}_3) A_{3[ia_0 \dots a_4]} \right. \\ & \left. - \mathcal{P}(z_2, \bar{z}_3) A_{4[ia_0 \dots a_4]} + \mathcal{P}(z_3, \bar{z}_3) A_{5[ia_0 \dots a_4]} - \mathcal{P}(\bar{z}_2, \bar{z}_3) A_{6[ia_0 \dots a_4]} \right) \end{aligned} \quad (52)$$

where $\mathcal{P}(z_i, z_j)$ is given in (19) and

$$\begin{aligned} A_{1[ia_0 \dots a_4]} &= (p_2 \cdot \varepsilon_3 \cdot D)_{[a_4} (\varepsilon_2 \cdot D)_{a_2 a_3} (p_2 \cdot D)_{a_0} (p_3)_i (p_3 \cdot D)_{a_1]} \\ A_{2[ia_0 \dots a_4]} &= (p_2 \cdot D \cdot \varepsilon_3 \cdot D)_{[a_4} (\varepsilon_2 \cdot D)_{a_2 a_3} (p_2)_i (p_3)_{a_0} (p_3 \cdot D)_{a_1]} \\ A_{3[ia_0 \dots a_4]} &= (p_3 \cdot D \cdot \varepsilon_3 \cdot D)_{[a_4} (\varepsilon_2 \cdot D)_{a_2 a_3} (p_2)_i (p_2 \cdot D)_{a_1} (p_3)_{a_0]} \\ A_{4[ia_0 \dots a_4]} &= (p_2 \cdot D \cdot \varepsilon_3^T)_{[a_4} (\varepsilon_2 \cdot D)_{a_2 a_3} (p_2 \cdot D)_{a_0} (p_3)_i (p_3 \cdot D)_{a_1]} \\ A_{5[ia_0 \dots a_4]} &= (p_3 \cdot D \cdot \varepsilon_3^T)_{[a_4} (\varepsilon_2 \cdot D)_{a_2 a_3} (p_2)_i (p_2 \cdot D)_{a_0} (p_3 \cdot D)_{a_1]} \\ A_{6[ia_0 \dots a_4]} &= (p_2 \cdot \varepsilon_3^T)_{[a_4} (\varepsilon_2 \cdot D)_{a_2 a_3} (p_2)_i (p_3)_{a_0} (p_3 \cdot D)_{a_1]} \end{aligned}$$

These factors are zero for symmetric polarization tensor ε_2 .

Using the fact that above kinematic factors contract with $\varepsilon_1^i \epsilon^{a_0 \dots a_4}$, one observes that there are 6 different terms in each case, however, 4 of them are zero and the other two terms are equal. They simplify as

$$\begin{aligned} A_{1[ia_0 \dots a_4]} &= \frac{1}{3} (p_2 \cdot \varepsilon_3)_{a_4} (\varepsilon_2)_{a_2 a_3} (p_2)_{a_0} (p_3)_i (p_3)_{a_1} \\ A_{2[ia_0 \dots a_4]} &= -\frac{1}{3} (p_2 \cdot D \cdot \varepsilon_3)_{a_4} (\varepsilon_2)_{a_2 a_3} (p_2)_{a_0} (p_3)_i (p_3)_{a_1} \\ A_{3[ia_0 \dots a_4]} &= \frac{1}{3} (p_3 \cdot D \cdot \varepsilon_3)_{a_4} (\varepsilon_2)_{a_2 a_3} (p_2)_i (p_2)_{a_1} (p_3)_{a_0} \\ A_{4[ia_0 \dots a_4]} &= \frac{1}{3} (p_2 \cdot D \cdot \varepsilon_3^T)_{a_4} (\varepsilon_2)_{a_2 a_3} (p_2)_{a_0} (p_3)_i (p_3)_{a_1} \\ A_{5[ia_0 \dots a_4]} &= -\frac{1}{3} (p_3 \cdot D \cdot \varepsilon_3^T)_{a_4} (\varepsilon_2)_{a_2 a_3} (p_2)_i (p_2)_{a_1} (p_3)_{a_0} \end{aligned}$$

$$A_{6[i a_0 \dots a_4]} = -\frac{1}{3}(p_2 \cdot \varepsilon_3^T)_{a_4}(\varepsilon_2)_{a_2 a_3}(p_2)_{a_0}(p_3)_i(p_3)_{a_1}$$

All terms in (47) produce structure $p_2 \cdot N \cdot \varepsilon_1$. On the other hand, only A_3, A_5 in the above amplitude produce this structure. The other terms produce $p_3 \cdot N \cdot \varepsilon_1$ structure. Since there is no momentum conservation in the transverse subspace the $p_2 \cdot N \cdot \varepsilon_1$ and $p_3 \cdot N \cdot \varepsilon_1$ are independent. The A_3, A_5 in the above amplitude have contribution to the $p_3 \cdot V \cdot \varepsilon_3$ structure of the amplitudes (47). Adding them, one finds that the $p_3 \cdot V \cdot \varepsilon_3$ structure of the amplitude (16) is zero for symmetric tensor ε_3 , and for antisymmetric tensor it is

$$\mathcal{A}(p_3 \cdot V \cdot \varepsilon_3, p_2 \cdot N \cdot \varepsilon_1) \sim 32\varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_3 \cdot V \cdot \varepsilon_3)_{a_3}(\varepsilon_2)_{a_1 a_4}(p_2)_i(p_2)_{a_2}(p_3)_{a_0} \mathcal{I}_4 \quad (53)$$

where

$$\mathcal{I}_4 = \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}}^2 \mathcal{P}(z_3, \bar{z}_3) K}{|z_{21}|^2 |z_{31}|^2 |z_{2\bar{1}}|^2 |z_{3\bar{1}}|^2} \quad (54)$$

The other terms in (52) produce structure $p_3 \cdot N \cdot \varepsilon_1$. They are

$$\begin{aligned} \mathcal{A}(p_2 \cdot V \cdot \varepsilon_3, p_3 \cdot N \cdot \varepsilon_1) &\sim 4\varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_2 \cdot V \cdot \varepsilon_3)_{a_4}(\varepsilon_2)_{a_2 a_3}(p_2)_{a_0}(p_3)_i(p_3)_{a_1} \mathcal{I}_5 \\ \mathcal{A}(p_2 \cdot N \cdot \varepsilon_3, p_3 \cdot N \cdot \varepsilon_1) &\sim 4\varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_2 \cdot N \cdot \varepsilon_3)_{a_4}(\varepsilon_2)_{a_2 a_3}(p_2)_{a_0}(p_3)_i(p_3)_{a_1} \mathcal{I}_6 \end{aligned} \quad (55)$$

where

$$\begin{aligned} \mathcal{I}_5 &= \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}}^2 K}{|z_{21}|^2 |z_{31}|^2 |z_{2\bar{1}}|^2 |z_{3\bar{1}}|^2} [\mathcal{P}(z_2, z_3) \pm \mathcal{P}(\bar{z}_2, \bar{z}_3) - \mathcal{P}(\bar{z}_2, z_3) \mp \mathcal{P}(z_2, \bar{z}_3)] \\ \mathcal{I}_6 &= \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}}^2 K}{|z_{21}|^2 |z_{31}|^2 |z_{2\bar{1}}|^2 |z_{3\bar{1}}|^2} [\mathcal{P}(z_2, z_3) \pm \mathcal{P}(\bar{z}_2, \bar{z}_3) + \mathcal{P}(\bar{z}_2, z_3) \pm \mathcal{P}(z_2, \bar{z}_3)] \end{aligned}$$

where the first sign is for symmetric and the second sign is for the antisymmetric polarization tensor ε_3 . Using the property of $\mathcal{P}(z_i, z_j) = -(\mathcal{P}(\bar{z}_i, \bar{z}_j))^*$, one finds that the integrand is pure imaginary for the symmetric case and is real for the antisymmetric case. The integral for the pure imaginary case is zero. By mapping the integral to unit disk and fixing the $SL(2, R)$ symmetry as in (32) one can show that the real integrals are the real integrals that appear in amplitudes (50) and (51), *i.e.*,

$$\mathcal{I}_5 = -2\mathcal{I}_2, \quad \mathcal{I}_6 = 2\mathcal{I}_3$$

Moreover, the integrand in \mathcal{I}_2 transforms to the integrand in \mathcal{I}_3 under $(z_2, \bar{z}_2, z_3, \bar{z}_3) \rightarrow (z_3, \bar{z}_3, z_2, \bar{z}_2)$, which means

$$\mathcal{I}_3(p_1, p_2, p_3) = \mathcal{I}_2(p_1, p_3, p_2) \quad (56)$$

The RR tensor in (55) can be written in terms of $F_{a_0 i}$, so they are invariant under the RR gauge transformation. The ε_2 can also be written in terms of the field strength. So they satisfy the Ward identity associated with ε_2 .

The 6 contractions in b_{10} in which two momenta contract with each other are

$$\begin{aligned} \mathcal{A}_{10}(pp) \sim & 2 \frac{6!}{5!} \varepsilon_1^i \epsilon^{a_0 \dots a_4} \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{11}^2 K}{|z_{21}|^2 |z_{31}|^2 |z_{2\bar{1}}|^2 |z_{3\bar{1}}|^2} \\ & \left(-\mathcal{P}(z_2, z_3) A_{7[ia_0 \dots a_4]} + \mathcal{P}(z_2, \bar{z}_2) A_{8[ia_0 \dots a_4]} + \mathcal{P}(\bar{z}_2, z_3) A_{9[ia_0 \dots a_4]} \right. \\ & \left. - \mathcal{P}(z_2, \bar{z}_3) A_{10[ia_0 \dots a_4]} + \mathcal{P}(z_3, \bar{z}_3) A_{11[ia_0 \dots a_4]} - \mathcal{P}(\bar{z}_2, \bar{z}_3) A_{12[ia_0 \dots a_4]} \right) \end{aligned} \quad (57)$$

where

$$\begin{aligned} A_{7[ia_0 \dots a_4]} &= p_2 \cdot p_3 (p_2 \cdot D)_{[i} (p_3 \cdot D)_{a_0} (\varepsilon_2 \cdot D)_{a_1 a_2} (\varepsilon_3 \cdot D)_{a_3 a_4]} \\ A_{8[ia_0 \dots a_4]} &= p_2 \cdot D \cdot p_2 (p_3)_{[i} (p_3 \cdot D)_{a_0} (\varepsilon_2 \cdot D)_{a_1 a_2} (\varepsilon_3 \cdot D)_{a_3 a_4]} \\ A_{9[ia_0 \dots a_4]} &= p_2 \cdot D \cdot p_3 (p_2)_{[i} (p_3 \cdot D)_{a_0} (\varepsilon_2 \cdot D)_{a_1 a_2} (\varepsilon_3 \cdot D)_{a_3 a_4]} \\ A_{10[ia_0 \dots a_4]} &= p_2 \cdot D \cdot p_3 (p_3)_{[i} (p_2 \cdot D)_{a_0} (\varepsilon_2 \cdot D)_{a_1 a_2} (\varepsilon_3 \cdot D)_{a_3 a_4]} \\ A_{11[ia_0 \dots a_4]} &= p_3 \cdot D \cdot p_3 (p_2)_{[i} (p_2 \cdot D)_{a_0} (\varepsilon_2 \cdot D)_{a_1 a_2} (\varepsilon_3 \cdot D)_{a_3 a_4]} \\ A_{12[ia_0 \dots a_4]} &= p_2 \cdot p_3 (p_2)_{[i} (p_3)_{a_0} (\varepsilon_2 \cdot D)_{a_1 a_2} (\varepsilon_3 \cdot D)_{a_3 a_4]} \end{aligned}$$

It is easy to verify that the above amplitude is zero when both NSNS polarization tensors are symmetric. When one of them is symmetric and the other one is antisymmetric, one finds some non-zero terms. However, the integrand for those terms are pure imaginary which are zero after integration. When both the NSNS polarization tensors are antisymmetric the above factors simplify to

$$\begin{aligned} A_{7[ia_0 \dots a_4]} &= -\frac{p_2 \cdot p_3}{6} [(p_2)_i (p_3)_{a_0} - (p_2)_{a_0} (p_3)_i] (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \\ A_{8[ia_0 \dots a_4]} &= \frac{p_2 \cdot D \cdot p_2}{3} (p_3)_i (p_3)_{a_0} (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \\ A_{9[ia_0 \dots a_4]} &= \frac{p_2 \cdot D \cdot p_3}{6} [(p_2)_i (p_3)_{a_0} + (p_2)_{a_0} (p_3)_i] (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \\ A_{10[ia_0 \dots a_4]} &= \frac{p_2 \cdot D \cdot p_3}{6} [(p_2)_i (p_3)_{a_0} + (p_2)_{a_0} (p_3)_i] (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \\ A_{11[ia_0 \dots a_4]} &= \frac{p_3 \cdot D \cdot p_3}{3} (p_2)_i (p_2)_{a_0} (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \\ A_{12[ia_0 \dots a_4]} &= \frac{p_2 \cdot p_3}{6} [(p_2)_i (p_3)_{a_0} - (p_2)_{a_0} (p_3)_i] (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \end{aligned}$$

Replacing them in (57), one finds the following structures:

$$\begin{aligned} \mathcal{A}(p_2 p_3) &\sim 2 \varepsilon_1^i \epsilon^{a_0 \dots a_4} [(p_2 \cdot V \cdot p_3) (p_2)_i (p_3)_{a_0} - (p_2 \cdot N \cdot p_3) (p_2)_{a_0} (p_3)_i] (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \mathcal{I}_6 \\ &\quad + 2 \varepsilon_1^i \epsilon^{a_0 \dots a_4} [(p_2 \cdot N \cdot p_3) (p_2)_i (p_3)_{a_0} - (p_2 \cdot V \cdot p_3) (p_2)_{a_0} (p_3)_i] (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \mathcal{I}_5 \\ \mathcal{A}(p_2 \cdot D \cdot p_2) &\sim 4 (p_2 \cdot D \cdot p_2) \varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_3)_i (p_3)_{a_0} (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \mathcal{I}_7 \\ \mathcal{A}(p_3 \cdot D \cdot p_3) &\sim 4 (p_3 \cdot D \cdot p_3) \varepsilon_1^i \epsilon^{a_0 \dots a_4} (p_2)_i (p_2)_{a_0} (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \mathcal{I}_4 \end{aligned} \quad (58)$$

where

$$\mathcal{I}_7 = \int d^2 z_1 d^2 z_2 d^2 z_3 \frac{z_{1\bar{1}}^2 \mathcal{P}(z_2, \bar{z}_2) K}{|z_{21}|^2 |z_{31}|^2 |z_{2\bar{1}}|^2 |z_{3\bar{1}}|^2}$$

Comparing the integrand in this integral with the integrand in \mathcal{I}_4 in (54), one finds the relation

$$\mathcal{I}_7(p_1, p_2, p_3) = \mathcal{I}_4(p_1, p_3, p_2) \quad (59)$$

We have seen that the amplitudes in equations (48), (50), (51), (53) and (55) are invariant under the RR and ε_2 gauge transformations. Hence, the amplitudes in (58) must also be invariant under these transformations. The terms which have $(p_2)_{a_0}$ are obviously invariant under the ε_2 gauge transformation. The terms which have $(p_3)_{a_0}$ are invariant under the ε_3 gauge transformation which should be considered in the $(2 \leftrightarrow 3)$ part of the amplitude (16). To check that the above amplitudes are invariant under the RR gauge transformation we have to know the relations between the integrals. Likewise, the combination of the above amplitudes and the amplitudes in (48), (50), (51), (53) and (55) must be invariant under the ε_3 gauge transformation provided that there is a relation between the integrals. We find these relations by demanding that the amplitude must be invariant under these transformations and then verify them by explicit calculation of the integrals for the restricted case of (38). The total amplitude is

$$\begin{aligned} \mathcal{A} \sim & 8(\varepsilon_1)_i \varepsilon_{a_0 \dots a_4} \left(-2(p_1 \cdot N \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_0} \mathcal{I}_1 + 4(p_3 \cdot V \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_0} \mathcal{I}_4 + (p_3 \cdot V \cdot p_3) p_2^i \varepsilon_3^{a_3 a_0} \mathcal{I}_4 \right. \\ & + (p_2 \cdot N \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_0} \mathcal{I}_2 + (p_2 \cdot V \cdot \varepsilon_3)^{a_3} p_3^i p_3^{a_0} \mathcal{I}_2 + \frac{1}{2} (p_2 \cdot V \cdot p_3) p_3^i \varepsilon_3^{a_3 a_0} \mathcal{I}_2 \\ & \left. - (p_2 \cdot V \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_0} \mathcal{I}_3 - (p_2 \cdot N \cdot \varepsilon_3)^{a_3} p_3^i p_3^{a_0} \mathcal{I}_3 - \frac{1}{2} (p_2 \cdot N \cdot p_3) p_3^i \varepsilon_3^{a_3 a_0} \mathcal{I}_3 \right) p_2^{a_2} \varepsilon_2^{a_1 a_4} + (2 \leftrightarrow 3) \end{aligned} \quad (60)$$

In the special case that the Mandelstam variables are constrained as (38), the above amplitude satisfies the Ward identity corresponding to ε_3 provided that there is the following relation between \mathcal{I}_1 and \mathcal{I}_4 :

$$p_3 \cdot V \cdot p_3 \mathcal{I}_4 = p_1 \cdot N \cdot p_3 \mathcal{I}_1 \quad (61)$$

Similar relation in the $(2 \leftrightarrow 3)$ part, *i.e.*, $p_2 \cdot V \cdot p_2 \mathcal{I}_7 = p_1 \cdot N \cdot p_2 \mathcal{I}_1$. Using these relations, one observes that the terms in (58) are invariant under the RR gauge transformation: The first two lines in (58) are zero under the constraint (38), and using conservation of momentum along the brane, the other terms can be write as

$$\begin{aligned} \mathcal{A}(p_2 \cdot D \cdot p_2) + \mathcal{A}(p_3 \cdot D \cdot p_3) \sim & -4(p_2 \cdot D \cdot p_2) \varepsilon_1^i \varepsilon^{a_0 \dots a_4} (p_3)_i (p_1)_{a_0} (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \mathcal{I}_7 \\ & -8[(p_1 \cdot N \cdot p_2)(\varepsilon_1 \cdot N \cdot p_3) - (p_1 \cdot N \cdot p_3)(\varepsilon_1 \cdot N \cdot p_2)] \varepsilon^{a_0 \dots a_4} (p_2)_{a_0} (\varepsilon_2)_{a_1 a_2} (\varepsilon_3)_{a_3 a_4} \mathcal{I}_1 \end{aligned}$$

The first term can be written in terms of field strength F_{ia_0} and the terms in the second line become zero upon replacing the RR polarization with its momentum.

In the general case, the last two terms in the second and the third lines of (60) each satisfies the Ward identity. The sum of the other terms then must satisfy the Ward identity. This happens if there are the following relations between the integrals:

$$-2p_1 \cdot N \cdot p_3 \mathcal{I}_1 + 2p_3 \cdot V \cdot p_3 \mathcal{I}_4 + p_2 \cdot N \cdot p_3 \mathcal{I}_2 - p_2 \cdot V \cdot p_3 \mathcal{I}_3 = 0 \quad (62)$$

From the $(2 \leftrightarrow 3)$ part, one finds the following relation:

$$-2p_1 \cdot N \cdot p_2 \mathcal{I}_1 + 2p_2 \cdot V \cdot p_2 \mathcal{I}_7 + p_2 \cdot N \cdot p_3 \mathcal{I}_3 - p_2 \cdot V \cdot p_3 \mathcal{I}_2 = 0 \quad (63)$$

Using the above two relations, one can show that the terms in (58) satisfy the Ward identity corresponding to the RR gauge transformation.

One can use the identities (62) and (63) to write the amplitude (60) in terms of RR field strength as

$$\begin{aligned} \mathcal{A} \sim & 8\epsilon_{a_0 \dots a_4} (F_1)_{a_0 i} \left(-2(p_1 \cdot N \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_2} \mathcal{I}_1 + 4(p_3 \cdot V \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_2} \mathcal{I}_4 + (p_2 \cdot N \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_2} \mathcal{I}_2 \right. \\ & \left. + (p_2 \cdot V \cdot \varepsilon_3)^{a_3} p_3^i p_3^{a_2} \mathcal{I}_2 - (p_2 \cdot V \cdot \varepsilon_3)^{a_3} p_2^i p_3^{a_2} \mathcal{I}_3 - (p_2 \cdot N \cdot \varepsilon_3)^{a_3} p_3^i p_3^{a_2} \mathcal{I}_3 \right) \varepsilon_2^{a_1 a_4} + (2 \leftrightarrow 3) \\ & -8\epsilon_{a_0 \dots a_4} (F_1)_{a_0 i} \left(p_3 \cdot V \cdot p_3 p_2^i \varepsilon_3^{a_3 a_2} \mathcal{I}_4 + \frac{1}{2} p_2 \cdot V \cdot p_3 p_3^i \varepsilon_3^{a_3 a_2} \mathcal{I}_2 - \frac{1}{2} p_2 \cdot N \cdot p_3 p_3^i \varepsilon_3^{a_3 a_2} \mathcal{I}_3 \right) \varepsilon_2^{a_1 a_4} \\ & + 8\epsilon_{a_0 \dots a_4} (F_1)_{ij} p_2^i p_3^j \varepsilon_2^{a_3 a_0} \varepsilon_3^{a_1 a_4} p_3^{a_2} \mathcal{I}_1 \end{aligned} \quad (64)$$

As a double check, we have calculated the amplitude (12) in the $(-1/2, -1/2)$ -picture and found exact agreement with the above result.

Using the relation (62) and the following identity:

$$\epsilon_{a_0 \dots a_4} p_\mu H_3^{\mu a_3 a_0} = \epsilon_{a_0 \dots a_4} (2p_\mu \varepsilon_3^{\mu a_3} p_3^{a_0} + (p_\mu p_3^\mu) \varepsilon_3^{a_3 a_0})$$

where p_μ is any vector, one can write the amplitude (60) in terms of field strength H_2, H_3 ,

$$\begin{aligned} \mathcal{A} \sim & \frac{4}{3} \epsilon_{a_0 \dots a_4} (\varepsilon_1)_i \left([2(p_1 \cdot N \cdot H_3)^{a_3 a_0} p_2^i \mathcal{I}_1 + 4(p_3 \cdot V \cdot H_3)^{a_3 a_0} p_2^i \mathcal{I}_4 \right. \\ & \left. + (p_2 \cdot N \cdot H_3)^{a_3 a_0} p_2^i \mathcal{I}_2 + (p_2 \cdot V \cdot H_3)^{a_3 a_0} p_3^i \mathcal{I}_2] H_2^{a_1 a_4 a_2} \right. \\ & \left. - [(p_3 \cdot N \cdot H_2)^{a_3 a_0} p_2^i \mathcal{I}_2 + (p_3 \cdot V \cdot H_2)^{a_3 a_0} p_3^i \mathcal{I}_2] H_3^{a_1 a_4 a_2} \right) + (2 \leftrightarrow 3) \end{aligned} \quad (65)$$

where we have also used the relations (49) and (56).

Mapping the integrand in each of the integrals $\mathcal{I}_2, \mathcal{I}_4$ to unit disk and fixing the $SL(2, R)$ symmetry as in (32), one finds that the integral over θ gives zero result when \mathcal{I}_2 is pure imaginary. The real integrals are

$$\begin{aligned} \mathcal{I}_2 &= 2 \int_0^1 dr_2 \int_0^1 dr_3 \frac{(1 - r_2^2)}{r_2} \int_0^{2\pi} d\theta \frac{[r_3(1 + r_2^2) - r_2(1 + r_3^2) \cos(\theta)] \widetilde{K}}{|1 - r_2 r_3 e^{i\theta}|^2 |r_2 - r_3 e^{-i\theta}|^2} \\ \mathcal{I}_4 &= - \int_0^1 dr_2 \int_0^1 dr_3 \frac{(1 + r_3^2)}{r_2 r_3 (1 - r_3^2)} \int_0^{2\pi} d\theta \widetilde{K} \end{aligned} \quad (66)$$

Up to this point the result is valid for the general case. However, to be able to perform the integrals we restrict the Mandelstam variables to (38). This restriction makes the θ -integral to be simple, *i.e.*,

$$\begin{aligned}\mathcal{I}_2 &= 4\pi \int_0^1 dr_3 \int_0^{r_3} dr_2 \frac{\widetilde{K}}{r_2 r_3} \\ \mathcal{I}_4 &= -2\pi \int_0^1 dr_2 \int_0^1 dr_3 \frac{(1+r_3^2)\widetilde{K}}{r_2 r_3 (1-r_3^2)}\end{aligned}\quad (67)$$

We have used the Maple to evaluate the θ -integral in \mathcal{I}_2 . Using the definition of the beta function, the radial integrals in \mathcal{I}_4 gives two beta functions, *i.e.*,

$$\mathcal{I}_4 = -\pi \frac{p_1 \cdot N \cdot p_3}{p_3 \cdot D \cdot p_3} B(p_1 \cdot p_2, 1 + p_2 \cdot D \cdot p_2) B(p_1 \cdot p_3, 1 + p_3 \cdot D \cdot p_3) \quad (68)$$

Using the relations (41) and (49), one observes that $\mathcal{I}_1, \mathcal{I}_4$ satisfy the relation (61) as expected.

Using the formula in the Appendix, one finds the radial integrals in \mathcal{I}_2 to be

$$\mathcal{I}_2 = \pi \frac{B(p_1 \cdot p_2 + p_1 \cdot p_3, 1 + p_3 \cdot D \cdot p_3)}{p_1 \cdot p_2} {}_3F_2 \left[\begin{matrix} p_1 \cdot p_2 + p_1 \cdot p_3, & p_1 \cdot p_2, & -p_2 \cdot D \cdot p_2 \\ 1 + p_1 \cdot p_2 + p_1 \cdot p_3 + p_3 \cdot D \cdot p_3, & 1 + p_1 \cdot p_2 \end{matrix} ; 1 \right]$$

To study the low energy limit of the amplitude (65), we expand $\mathcal{I}_2, \mathcal{I}_4$ at the low energy. The expansion of beta function is standard and for expanding the hypergeometric function we use the package [22]. The result is

$$\begin{aligned}\mathcal{I}_2 &= \pi \left(\frac{1}{p_1 \cdot p_2 (p_1 \cdot p_2 + p_1 \cdot p_3)} - \frac{\pi^2}{6} \frac{p_3 \cdot D \cdot p_3}{p_1 \cdot p_2} + \dots \right) \\ \mathcal{I}_4 &= -\pi \left(\frac{1}{2p_1 \cdot p_2 p_1 \cdot p_3} + \frac{1}{p_1 \cdot p_2 p_3 \cdot D \cdot p_3} - \frac{\pi^2}{12} \left[\frac{2p_1 \cdot N \cdot p_3}{p_1 \cdot p_2} + \frac{2p_2 \cdot D \cdot p_2}{p_3 \cdot D \cdot p_3} + \frac{p_2 \cdot D \cdot p_2}{p_1 \cdot p_3} \right] + \dots \right)\end{aligned}$$

From these expansions and the expansion (42) for \mathcal{I} , one realizes that only \mathcal{I}_4 has massless open string pole at order $O(\alpha'^2)$. Hence, the amplitude (65) has the following massless open string pole at order $O(\alpha'^2)$:

$$\mathcal{A}^f \sim \frac{8\pi^3}{9} \frac{p_2 \cdot D \cdot p_2}{p_3 \cdot D \cdot p_3} \epsilon_{a_0 \dots a_4} (\varepsilon_1)_i (p_3 \cdot V \cdot H_3)^{a_3 a_0} p_2^i H_2^{a_1 a_4 a_2} + (2 \leftrightarrow 3)$$

This is exactly the same as the amplitude (11) in field theory side. This ends our illustration of the consistency of the couplings in (4) with the string theory S-matrix element of one RR and two NSNS states.

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Appendix

In this appendix we calculate the following integral:

$$I = \int_0^1 dx \int_0^x dy x^a y^b (1-x)^c (1-y)^d \quad (69)$$

To take the integral, one should first change the variable to $y = xu$, *i.e.*,

$$I = \int_0^1 dx x^{1+a+b} (1-x)^c \int_0^1 du u^b (1-xu)^d \quad (70)$$

Then using the integral representation of the generalized hypergeometric function ${}_pF_q$ [23]:

$${}_2F_1 \left[\begin{matrix} -c, 1+a \\ 2+a+b \end{matrix} ; x \right] B(1+a, 1+b) = \int_0^1 du u^a (1-u)^b (1-xu)^c$$

$${}_{p+1}F_{q+1} \left[\begin{matrix} 1+a, a_1, \dots, a_p \\ 2+a+b, b_1, \dots, b_q \end{matrix} ; \lambda \right] B(1+a, 1+b) = \int_0^1 dx x^a (1-x)^b {}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; \lambda \right]$$

one finds

$$I = \frac{B(1+c, 2+a+b)}{1+b} {}_3F_2 \left[\begin{matrix} 2+a+b, 1+b, -d \\ 3+a+b+c, 2+b \end{matrix} ; 1 \right] \quad (71)$$

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